Ellipses of constant entropy in the *XY* spin chain

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F Franchini[1,2], **A R Its**[3], **B-Q Jin**[4] **and V E Korepin**[5]

[1] Department of Physics and Astronomy, State University of New York at Stony Brook, Stony Brook, NY 11794, USA
[2] The Abdus Salam ICTP, Strada Costiera 11, Trieste 34100, Italy
[3] Department of Mathematical Sciences, Indiana University–Purdue University Indianapolis, Indianapolis, IN 46202-3216, USA
[4] College of Physics and Electronic Information, Wenzhou University, Wenzhou, Zhejiang, People's Republic of China
[5] CN Yang Institute for Theoretical Physics, State University of New York at Stony Brook, Stony Brook, NY 11794-3840, USA

E-mail: fabio@ictp.it, itsa@math.iupui.edu, jinbq@wzu.edu.cn and korepin@insti.physics.sunysb.edu

Abstract
Entanglement in the ground state of the XY model on the infinite chain can be measured by the von Neumann entropy of a block of neighbouring spins. We study a double scaling limit: the size of the block is much larger than 1 but much smaller than the length of the whole chain. The entropy of the block has an asymptotic limit in the gapped regimes. We study this limiting entropy as a function of the anisotropy and of the magnetic field. We identify its minima at product states and its divergencies at the quantum phase transitions. We find that the curves of constant entropy are ellipses and hyperbolas, and that they all meet at one point (*essential critical point*). Depending on the approach to the essential critical point, the entropy can take any value between 0 and ∞. In the vicinity of this point, small changes in the parameters cause large change of the entropy.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

Entanglement is a primary resource for quantum information processing [1–4]. It represents the ability of using purely quantum effects to control one system by another. Stable and large scale entanglement is necessary for scalability of quantum computation. In [3], the von Neumann entropy of a subsystem was proposed as a measure of its entanglement. Since then, essential progresses have been achieved in the understanding of entanglement as applied to

various quantum systems. One of them was the observation made in [5] that for spin chains the entropy of a large block of spins behaves differently for gapless and gapfull models. Singularity of the entropy at phase transitions was previously discovered in [6]. In many dimensions, entropy scales as the area of the boundary [7–11]. For one-dimensional systems, the entropy was calculated using conformal field theory methods in [12] and it was confirmed that for gapless models it scales logarithmically with the size of the block. This behaviour was related to the second law of thermodynamics in [13]. One should also note that the entropy of a large block of spins behaves differently in the ferromagnetic and in the anti-ferromagnetic *XXX* [14]. For gapped theories, the entropy of a large block of spins reaches a limit, i.e. it saturates [5]. This was explicitly described for AKLT models in [15].

In this paper, we calculate the entanglement of the ground state of the *XY* model (on the infinite chain) by considering the von Neumann entropy of a block of L neighbouring spins in the limit $L \to \infty$. We use the results of [16–18] and extend them to the whole phase diagram of the model[6]. These results were derived using a determinant representation of entanglement introduced in [19, 20]. The entanglement of the Ising model, a subsystem of the *XY* model we consider here, was described in [21].

The *XY* model is a particularly important system for quantum information, for his analytical tractability on one side and for the possibility of physical realization on the other side. Optical lattices and other physical systems [22–24] can be described by this model and the former, in particular, are considered as very promising candidates as hardware constituents of quantum computers.

The Hamiltonian of the *XY* model is

$$\mathcal{H} = - \sum_{n=-\infty}^{\infty} \left[(1 + \gamma)\sigma_n^x \sigma_{n+1}^x + (1 - \gamma)\sigma_n^y \sigma_{n+1}^y + h\sigma_n^z \right]. \tag{1}$$

Here, $\gamma \geqslant 0$ is the anisotropy parameter, σ_n^x, σ_n^y and σ_n^z are the Pauli matrices and $h \geqslant 0$ is the magnetic field. The model is clearly symmetric under the transformation $\gamma \to -\gamma$ or $h \to -h$. In [16, 18] only the case $0 \leqslant \gamma \leqslant 1$ was discussed; here we can confirm that those results can be directly extended for $\gamma > 1$ by analytical continuation.

The *XY* model was solved in [25–28]. The methods of Toeplitz determinants and integrable Fredholm operators were used for the evaluation of correlation functions, see [27, 29–33]. The idea to use the determinants for the calculation of the entropy was put forward in [19].

The solution of the *XY* model looks differently in three cases:

Case 2 is defined by $h > 2$: this is a strong magnetic field.
Case 1A is defined by $h < 2$ and $\gamma > \sqrt{1 - (h/2)^2}$: moderate magnetic field for small anisotropy, and includes zero magnetic field for large anisotropy.
Case 1B is defined by $h < 2$ and $\gamma < \sqrt{1 - (h/2)^2}$: it describes weak magnetic field, including zero magnetic field in the small anisotropy regime.

At $\gamma = 0$ and for $h \leqslant 2$, the model is known as the isotropic *XY* model (or *XX* model) and its spectrum is gapless. The entropy for this critical phase was calculated in [19]. The boundary between cases 2 and 1A ($h = 2$) is also critical. In the rest phase diagram, the spectrum of the *XY* model is given by [26, 27]

$$\epsilon_k = 4\sqrt{(\cos k - h/2)^2 + \gamma^2 \sin^2 k}, \tag{2}$$

where $-\pi < k < \pi$. We draw the phase diagram and the three cases we are considering in figure 1.

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[6] Moreover, in the appendix we explain how these results can be used to calculate the entanglement of the *XY* model in a staggered magnetic field.
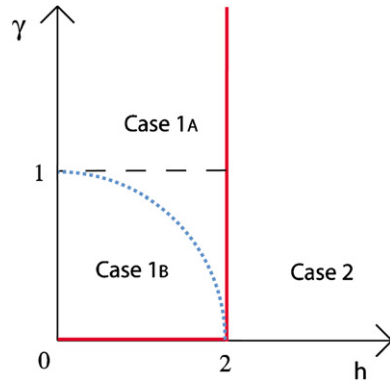
Figure 1. Phase diagram of the anisotropic XY model in a constant magnetic field (only $\gamma \geqslant 0$ and $h \geqslant 0$ shown). The three cases 2, 1A, 1B, considered in this paper, are clearly marked. The critical phases ($\gamma = 0, h \leqslant 2$ and $h = 2$) are drawn in bold lines (red, online). The boundary between cases 1A and 1B, where the ground state is given by two degenerate product states, is shown as a dotted line (blue, online). The Ising case ($\gamma = 1$) is also indicated as a dashed line.

At the boundary between cases 1A and 1B ($h = 2\sqrt{1 - \gamma^2}$), the ground state can be expressed as a *product state* as it was discovered in [34]. The ground state is in fact *doubly degenerated*:

$$|GS_1\rangle = \prod_{n \in \text{lattice}} [\cos(\theta)|\uparrow_n\rangle + \sin(\theta)|\downarrow_n\rangle],$$

$$|GS_2\rangle = \prod_{n \in \text{lattice}} [\cos(\theta)|\uparrow_n\rangle - \sin(\theta)|\downarrow_n\rangle]. \tag{3}$$

Here, $\cos^2(2\theta) = (1 - \gamma)/(1 + \gamma)$ and $|\uparrow_n\rangle, |\downarrow_n\rangle$ are the eigenstates of the operator σ_n^z at the nth lattice site. The role of factorized states such as these was emphasized in [29, 35–37]. Let us mention that even on this line the rest of the energy levels are separated by a gap and correlation functions decay exponentially. The boundary between cases 1A and 1B is not a phase transition.

2. Block entropy

In general, we denote the ground state of the model by $|GS\rangle$. We consider the entropy of a block of L neighbouring spins: it measures the entanglement between the block and the rest of the chain [3, 5]. We treat the whole ground state as a binary system $|GS\rangle = |A\&B\rangle$. The block of L neighbouring spins is subsystem A and the rest of the ground state is subsystem B. The density matrix of the ground state is $\rho_{AB} = |GS\rangle\langle GS|$. The density matrix of the block is $\rho_A = Tr_B(\rho_{AB})$. The entropy $S(\rho_A)$ of the block is

$$S(\rho_A) = -Tr_A(\rho_A \ln \rho_A). \tag{4}$$

Note that each of the ground states (3) *is factorized and has no entropy.*

To express the entropy we need the complete elliptic integral of the first kind,

$$I(k) = \int_0^1 \frac{dx}{\sqrt{(1 - x^2)(1 - k^2 x^2)}} \tag{5}$$

and the modulus

$$\tau_0 = I(k')/I(k), \qquad k' = \sqrt{1 - k^2}. \tag{6}$$

The magnetic field and anisotropy define the elliptic parameter k:

$$k = \begin{cases} \gamma/\sqrt{(h/2)^2 + \gamma^2 - 1}, & \text{case 2} \\ \sqrt{(h/2)^2 + \gamma^2 - 1}/\gamma, & \text{case 1A} \\ \sqrt{1 - \gamma^2 - (h/2)^2}/\sqrt{1 - (h/2)^2}, & \text{case 1B}, \end{cases} \tag{7}$$

where k vanishes at large magnetic fields ($h \to \infty$), at $\gamma = 0$ for $h > 2$ and at the boundary between cases 1A and 1B ($h = 2\sqrt{1 - \gamma^2}$). In all these regions of the phase diagram, the ground state of the system is given by product states (a ferromagnetic state in the first two cases and the doubly degenerate state (3) for the latter). At the phase transitions ($h = 2$ and $\gamma = 0$, $h < 2$), the elliptic parameter $k = 1$.

In paper [16], we used determinant representation for the evaluation of the entropy. The zeros of the determinant form an infinite sequence of numbers:

$$\lambda_m = \tanh\left[\left(m + \frac{1 - \sigma}{2}\right)\pi\tau_0\right], \tag{8}$$

where $\sigma = 1$ in case 1 and $\sigma = 0$ in case 2 and m is an integer. Note $0 < \lambda_m < 1$ and $\lambda_m \to 1$ as $m \to \infty$.

These zeros allowed us to represent the entropy as a convergent series in [16]:

$$S(\rho_A) = \sum_{m=-\infty}^{\infty} (1 + \lambda_m) \ln \frac{2}{1 + \lambda_m}. \tag{9}$$

Peshel also obtained series (9) in cases of the non-zero magnetic field, see [17].[7] He summed it up into

$$S(\rho_A) = \frac{1}{6}\left[\ln\frac{4}{kk'} + (k^2 - k'^2)\frac{2I(k)I(k')}{\pi}\right], \qquad \text{case 2,} \tag{10}$$

$$S(\rho_A) = \frac{1}{6}\left[\ln\left(\frac{k^2}{16k'}\right) + (2 - k^2)\frac{2I(k)I(k')}{\pi}\right] + \ln 2, \qquad \text{case 1A.} \tag{11}$$

In our paper [16], we have shown that equation (9) is valid for all three cases, which allowed us to sum up series (9) in the case of the weak magnetic field (including the zero magnetic field) as well:

$$S(\rho_A) = \frac{1}{6}\left[\ln\left(\frac{k^2}{16k'}\right) + (2 - k^2)\frac{2I(k)I(k')}{\pi}\right] + \ln 2, \qquad \text{case 1B,} \tag{12}$$

obtaining the same expression as in (11), but with a different definition of k as from (7). The rigorous proof and the precise history is given in paper [18]. In addition to what was reported there, here we can confirm that all these results are valid also for $\gamma > 1$.

3. Analysis of the entropy

Now we can study the range of variation of the limiting entropy. We find a *local minimum* $S(\rho_A) = \ln 2$ at the boundary between cases 1A and 1B ($h = 2\sqrt{1 - \gamma^2}$). This is the case of the doubly degenerated ground state (3) and it is consistent with [38], where it was shown that when the ground state becomes a superposition of two product states with different quantum numbers, then the entropy of a subsystem turns into $\ln 2$.

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[7] In comparing with the results of [17], the reader should keep in mind that Peshel calculated the entropy per boundary, therefore, his results differ by a factor of 2 compared to those in this paper.
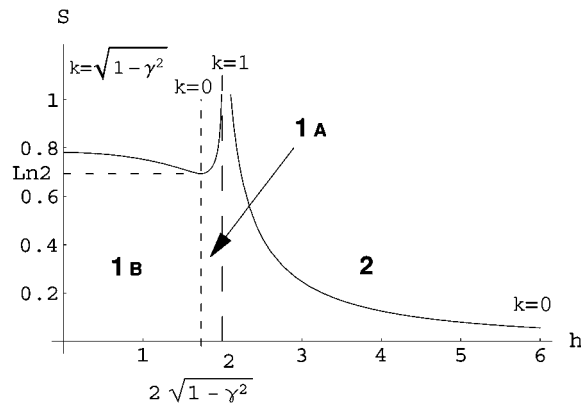
Figure 2. The limiting entropy as a function of the magnetic field at the constant anisotropy $\gamma = 1/2$. The entropy has a local minimum $S = \ln 2$ at $h = 2\sqrt{1 - \gamma^2}$ and the absolute minimum for $h \to \infty$ where it vanishes. S is singular at the phase transition $h = 2$ where it diverges to $+\infty$. The three cases are marked.
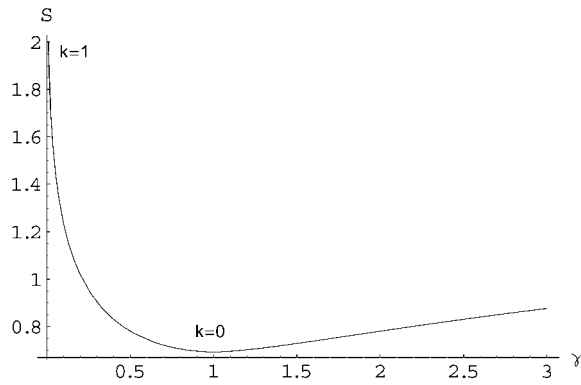

Figure 3. The limiting entropy as a function of the anisotropy parameter at the constant vanishing magnetic field $h = 0$. The entropy has a minimum $S = \ln 2$ at $\gamma = 1$ corresponding to the boundary between cases 1A and 1B. S diverges to $+\infty$ at the phase transition $\gamma = 0$.

The absolute minimum ($S = 0$) is achieved at infinite magnetic field or at $\gamma = 0$ for $h > 2$, where the ground state becomes ferromagnetic (i.e. all spins are parallel). The entropy diverges to $+\infty$, i.e. has singularities, at the *phase transitions*: $h = 2$ or $\gamma = 0$ and $h < 2$. To show this behaviour of the limiting entropy, we plot it as a function of the magnetic field h at a constant anisotropy $\gamma = 1/2$ in figure 2.

We provide a plot of the entropy as a function of γ at a constant (vanishing) magnetic field ($h = 0$) in figure 3, where the local minimum $S = \ln 2$ is again visible at $\gamma = 1$. Figure 4 is a three-dimensional plot of the entropy as a function of both the anisotropy parameter γ and the external magnetic field h, and all the features discussed so far are visible.

3.1. The Ising point

The degenerate product states case (3) is particularly interesting. For $h = 0$, $\gamma = 1$, the XY model reduces to the Ising model and the ground state (3) is given by the *Bell–Pair* states:
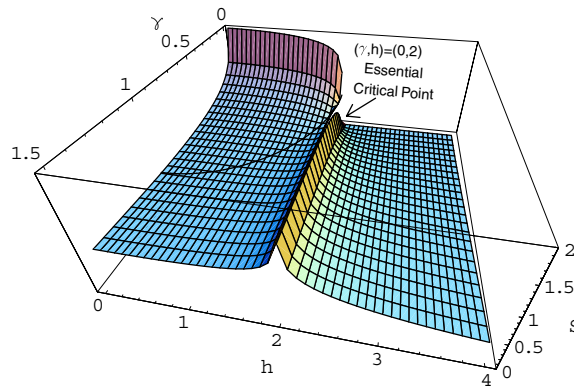
Figure 4. A three-dimensional plot of the limiting entropy as a function of the anisotropy parameter γ and the external magnetic field h. The local minimum $S = \ln 2$ at the boundary between cases 1A and 1B is visible and marked by a continuum line. S diverges to $+\infty$ at the phase transitions $h = 2$ and $\gamma = 0, h \leqslant 2$. The entropy takes every positive value in the vicinity of the essential critical point $(h, \gamma) = (2, 0)$.
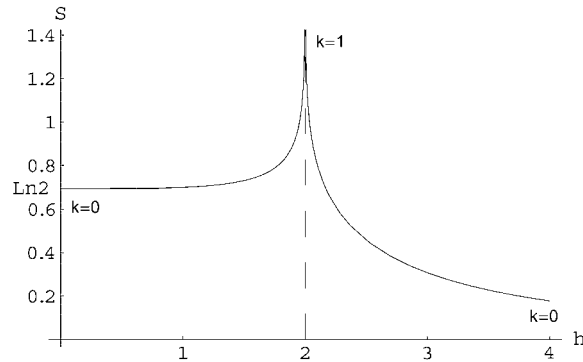

Figure 5. The limiting entropy as a function of the magnetic field at the Ising point $\gamma = 1$. The entropy has a local minimum $S = \ln 2$ at $h = 0$ and the absolute minimum for $h \to \infty$ where it vanishes. S is singular at the phase transition $h = 2$ where it diverges to $+\infty$.

$$|GS_1\rangle = \prod_{n \in \text{lattice}} \frac{1}{\sqrt{2}}(|\uparrow_n\rangle + |\downarrow_n\rangle),$$
$$|GS_2\rangle = \prod_{n \in \text{lattice}} \frac{1}{\sqrt{2}}(|\uparrow_n\rangle - |\downarrow_n\rangle). \tag{13}$$

We plot the entropy as a function of the magnetic field at the Ising point $\gamma = 1$ in figure 5. One can note that the local minimum $S = \ln 2$ is achieved at $h = 0$.

3.2. The essential critical point

Another interesting limit is reached around the point $\gamma = 0, h = 2$. This point belongs to both the critical phases of the XY model, so the entropy does not have an analytical expression (fixed value) on this point, but we can study the behaviour of $S(\rho_A)$ in the vicinity of this point. We already studied a couple of trajectories reaching this critical point: along the boundary between cases 1A and 1B ($h = 2\sqrt{1 - \gamma^2}$), the entropy is on its local minimum $S = \ln 2$.
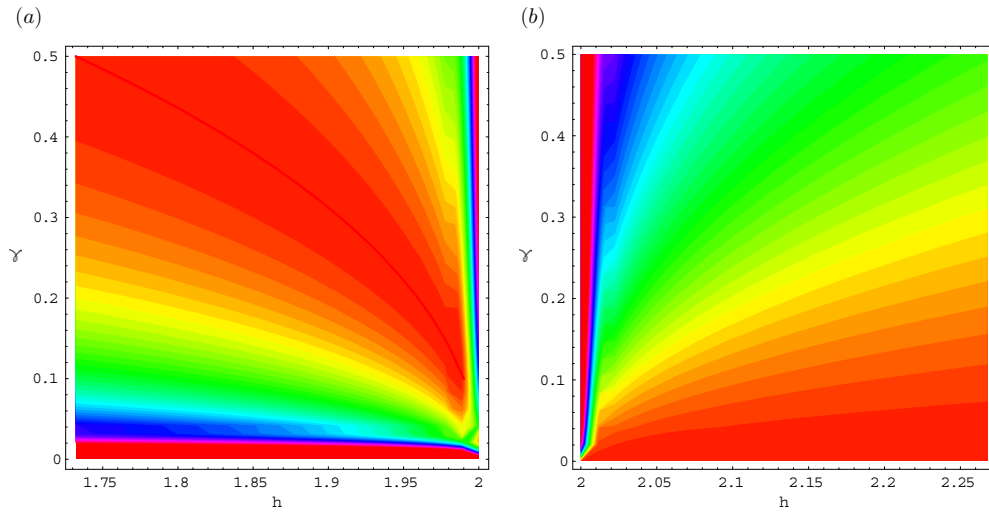
(a)

(b)

Figure 6. A contour plot of the limiting entropy near the essential critical point $h = 2, \gamma = 0$. Regions of similar colours have similar entropy values and the lines where colours change are the lines of constant entropy. $S(\rho_A)$ diverges to $+\infty$ on the critical lines $h = 2$ and $h < 2, \gamma = 0$. One can see that near the essential critical point the lines of constant entropy grow denser.

Along the critical lines ($h = 2$ and $\gamma = 0$ for $h < 2$) the entropy is divergent, while for $\gamma = 0$ and $h > 2$ the ground state is ferromagnetic and the entropy is 0. Since the limit of the entropy reaching the point $(h, \gamma) = (2, 0)$ does not exist (it is direction-dependent), we call this point the *essential critical point*. In the next section, we study the vicinity of this point and show that depending on the direction of approach the entropy can take any positive value. In figure 6, we present a contour plot of the entropy around the essential critical point. From this plot, one can see that the entropy can assume a wide range of values near the point.

4. Ellipses and hyperbolas of constant entropy

We now look for curves of constant entropy. Since the entropy depends only on the elliptic parameter (7), the curves of constant entropy are the curves of constant k. Such trajectories are easily found and the family of curves of constant entropy can be written in terms of a single parameter κ:

case 2 $\quad \left\{ h > 2: \qquad \left(\frac{h}{2}\right)^2 - \left(\frac{\gamma}{\kappa}\right)^2 = 1, \qquad 0 \leqslant \kappa < \infty \right. \qquad (14)$

case 1A $\quad \left\{ \begin{matrix} h < 2, \\ \gamma > \sqrt{1 - (h/2)^2} \end{matrix} \right. : \quad \left(\frac{h}{2}\right)^2 + \left(\frac{\gamma}{\kappa}\right)^2 = 1, \qquad \kappa > 1 \qquad (15)$

case 1B $\quad \left\{ \begin{matrix} h < 2, \\ \gamma < \sqrt{1 - (h/2)^2} \end{matrix} \right. : \quad \left(\frac{h}{2}\right)^2 + \left(\frac{\gamma}{\kappa}\right)^2 = 1, \qquad \kappa < 1. \qquad (16)$

For $h > 2$ (case 2), the curves of constant entropy are hyperbolas, while for $h < 2$ (cases 1A and 1B) they are ellipses.

Each point in the phase diagram of the XY model belongs to one of these curves. By selecting a value of the parameter κ, we select a family of points with the same elliptic parameter k in (7). There is a one-to-one correspondence between k and κ:

$$k = \sqrt{\frac{\kappa^2}{1 + \kappa^2}} \qquad k' = \sqrt{\frac{1}{1 + \kappa^2}} \qquad \text{case 2} \tag{17}$$

$$k = \sqrt{\frac{\kappa^2 - 1}{\kappa^2}} \qquad k' = \frac{1}{\kappa} \qquad \text{case 1A} \tag{18}$$

$$k = \sqrt{1 - \kappa^2} \qquad k' = \kappa \qquad \text{case 1B.} \tag{19}$$

We recognize that $\kappa = 1$, as the boundary between cases 1A and 1B is the curve where the ground state can be expressed as a doubly degenerate product state (3).

It is important to note that this curve has all the essential critical points $(h, \gamma) = (2, 0)$ in common. This means that starting from any point in the phase diagram of the XY model, one always reaches the essential critical point by following a curve of constant entropy.

For $h < 2$, the entropy has a minimum at $\ln 2$ and diverges to $+\infty$ approaching the critical line $h = 2$. For $h > 2$, $S(\rho_A)$ decreases monotonically from $+\infty$ near the critical line to 0 at infinite magnetic field. Beside the critical lines, the entropy is a continuous function, so its range is the positive real axis.

This means that, depending on the direction of approach, the entropy assumes every positive number near the essential critical point, since every ellipse or hyperbola of constant entropy passes through that point. In other words, a small deviation from the essential critical point can bring a big change in the value of the entropy. This is very important from the point of view of *quantum control* because it allows us to change dramatically the entanglement (and hence the quantum computing capabilities) with small changes in the parameters of the system.

It is easy to express the entropy in terms of the parameter κ defining the ellipses and hyperbolas of constant entropy:

$$S(\rho_A) = \frac{1}{6}\left[\ln\frac{4(\kappa^2 + 1)}{\kappa} + \frac{2}{\pi}\frac{\kappa^2 - 1}{\kappa^2 + 1} I\left(\sqrt{\frac{\kappa^2}{\kappa^2 + 1}}\right) I\left(\sqrt{\frac{1}{\kappa^2 + 1}}\right) \right], \qquad \text{case 2,}$$

$$S(\rho_A) = \frac{1}{6}\left[\ln\frac{\kappa^2 - 1}{16\kappa} + \frac{2}{\pi}\frac{(\kappa^2 + 1)}{\kappa^2} I\left(\sqrt{\frac{\kappa^2 - 1}{\kappa^2}}\right) I\left(\frac{1}{\kappa}\right) \right] + \ln 2, \qquad \text{case 1A,} \tag{20}$$

$$S(\rho_A) = \frac{1}{6}\left[\ln\frac{1 - \kappa^2}{16\kappa} + \frac{2}{\pi}(\kappa^2 + 1)I(\sqrt{1 - \kappa^2})I(\kappa) \right] + \ln 2, \qquad \text{case 1B.}$$

5. Entropy approaching the critical lines

Using the formulae of the previous section, we are now in a position to discuss the divergences of the entropy near the critical phases. General results exist in these cases based on a conformal field theory approach [12] and specific results were derived for the isotropic case ($\gamma = 0$) in [19]. We know that in the double-scaling limit we are considering, the entropy diverges logarithmically with the size of the block. The coefficient of this logarithmical divergence can be calculated by knowing the central charge of the corresponding conformal field theory at the critical point [12].

Setting $\kappa = 0$ or $\kappa = \infty$, the ellipses and hyperbolas of constant entropy (16) collapse into the critical lines, i.e. a vertical line at $\gamma = 0$ or a horizontal line at $h = 2$, respectively.

Table 1. Recap of the results with the entropy in different regions of the phase diagram, the curves (ellipses and hyperbolas) of constant entropy and the relationship between the elliptic parameter k and the parameter κ defining the family of curves.

Region	$S(\rho_A)$	Curves of constant S	Range of parameters
$2:\quad h > 2$	$\frac{1}{6}\left[\ln\frac{4}{kk'} + \frac{2(k^2-k'^2)I(k)I(k')}{\pi}\right]$	$\left(\frac{h}{2}\right)^2 - \left(\frac{\gamma}{\kappa}\right)^2 = 1$	$0 \leqslant k < 1$ $0 \leqslant \kappa < \infty$ $k = \sqrt{\frac{\kappa^2}{1+\kappa^2}}$
$1A : \begin{cases} h < 2, \\ \gamma > \sqrt{1-(h/2)^2} \end{cases}$	$\frac{1}{6}\left[\ln\frac{k^2}{16k'} + \frac{2(2-k^2)I(k)I(k')}{\pi}\right] + \ln 2$	$\left(\frac{h}{2}\right)^2 + \left(\frac{\gamma}{\kappa}\right)^2 = 1$	$0 < k < 1$ $\kappa > 1$ $k = \sqrt{\frac{\kappa^2-1}{\kappa^2}}$
$1B : \begin{cases} h < 2, \\ \gamma < \sqrt{1-(h/2)^2} \end{cases}$	$\frac{1}{6}\left[\ln\frac{k^2}{16k'} + \frac{2(2-k^2)I(k)I(k')}{\pi}\right] + \ln 2$	$\left(\frac{h}{2}\right)^2 + \left(\frac{\gamma}{\kappa}\right)^2 = 1$	$0 < k < 1$ $\kappa < 1$ $k = \sqrt{1-\kappa^2}$
$\gamma = \sqrt{1-(h/2)^2}$	$\ln 2$	$\left(\frac{h}{2}\right)^2 + \gamma^2 = 1$	$k = 0$ $\kappa = 1$

Using (20), we can study how the entropy diverges approaching these lines. Using case 1B in (20), we can take $\kappa \to 0$ to find

$$S(\kappa \to 0, h < 2) \sim -\frac{1}{3}\ln\left(\frac{\kappa}{2}\right) + \cdots$$
$$= -\frac{1}{3}\ln\left(\frac{\gamma}{2}\right) + \frac{1}{6}\ln[1 - (h/2)^2] + \cdots \tag{21}$$

which is consistent with the results obtained in [19] for the isotropic case ($\gamma = 0$).

We can investigate how the entropy approaches the critical line $h = 2$ from below and from above. In the former case, we shall set $\kappa \to \infty$ in case 1A of (20):

$$S(\kappa \to \infty, h < 2) \sim \frac{1}{3}\ln\left(\frac{\kappa}{2}\right) + \cdots$$
$$= -\frac{1}{6}\ln[1 - (h/2)^2] + \frac{1}{3}\ln\left(\frac{\gamma}{2}\right) + \cdots. \tag{22}$$

In the latter case, for a direction almost parallel to the critical line $h = 2$, but slightly above it, we take $\kappa \to \infty$ in case 2 of (20):

$$S(\kappa \to \infty, h > 2) \sim \frac{1}{3}\ln(4\kappa) + \cdots$$
$$= -\frac{1}{6}\ln[(h/2)^2 - 1] + \frac{1}{3}\ln(4\gamma) + \cdots. \tag{23}$$

These results are in agreement with the conclusion of [12].

6. Conclusions

We analysed the entanglement in the ground state of the XY model on the infinite chain by studying the von Neumann entropy $S(\rho_A)$ of a block A of neighbouring spins. This entropy is an effective measure of the quantum computing capabilities of a system and plays a fundamental role in the field of quantum information.

Using previously known results for the entropy in the limit of a large block of spins [16–18], we studied the behaviour of $S(\rho_A)$ in the phase diagram of the XY model (see table 1). We found that for $h < 2$, the entropy has a local minimum $S = \ln 2$ on the curve

$(h/2)^2 + \gamma^2 = 1$. On this line the ground state is a doubly degenerate linear combination of product states. The entropy diverges to $+\infty$ at the phase transitions $h = 2$ and $\gamma = 0, h < 2$. For $h > 2$, the entropy reaches the absolute minimum at infinite magnetic field $h \to +\infty$ and for $\gamma = 0$, i.e. when the ground state is ferromagnetic. $S(\rho_A)$ diverges to $+\infty$ on the critical line $h = 2$ and it is continuous otherwise.

We identified a set of curves (ellipses and hyperbolas) of constant entropy. They are given in (14)–(16). All these curves have one point in common that we decided to call the *essential critical point*: $(h, \gamma) = (2, 0)$. The fact that all the curves of constant entropy pass through one point, together with the fact that the range of the entropy as a function of γ and h is the positive real axis, means that the entropy can assume any real positive value near the essential critical point, depending on the direction of approach. In turn, this means that the essential critical point is very important for quantum control in that small changes in the parameters can change the entanglement dramatically.

With this work, we conclude the analysis of the asymptotic von Neumann entropy for the bi-partite one-dimensional XY model. We covered the whole phase diagram (including $\gamma > 1$) focusing on the sector $h \geqslant 0$ and $\gamma \geqslant 0$; since the model is invariant under the substitution $\gamma \to -\gamma$ or $h \to -h$, the results for the entanglement can be extended immediately to the negative values of the anisotropy parameter γ or of the magnetic field h.

Finally, we note that the work done so far on the XY model in a constant magnetic field allows us to calculate the bi-partite entropy of the XY model in a staggered magnetic field as well. As we discuss in the appendix, there is an exact mapping between these two models. Therefore, the knowledge of the entanglement for one of the models automatically gives the entanglement for the other. We give some details in the appendix.

Acknowledgments

Appendix. XY model in a staggered magnetic field

It is a well-known fact in the theory of integrable models that there is an exact mapping between the traditional XY model in the constant magnetic field described by Hamiltonian (1) and the XY model in the staggered magnetic field:

$$\mathcal{H}' = -J \sum_{n=-\infty}^{\infty} (1 + \gamma')\sigma_n^x \sigma_{n+1}^x + (1 - \gamma')\sigma_n^y \sigma_{n+1}^y + (-1)^n h' \, \sigma_n^z. \tag{A.1}$$

This mapping is achieved by performing a rotation of every other spin along the x-direction. To identify the two Hamiltonians, one also needs to substitute $\gamma \to 1/\gamma$ [8] and to rescale the magnetic field and the Hamiltonian by a factor of $1/\gamma$ and γ, respectively:

$$\gamma' = 1/\gamma, \qquad h' = h/\gamma, \qquad J = \gamma. \tag{A.2}$$

In the main body of this paper, we analysed the entanglement of the XY model in a constant magnetic field (1). The results we derived can be applied directly to calculate the bi-partite entanglement of the XY model in a staggered field (A.1). All formulae are valid and

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[8] Note that the large anisotropy regime is so mapped into the small anisotropy regime and vice versa.
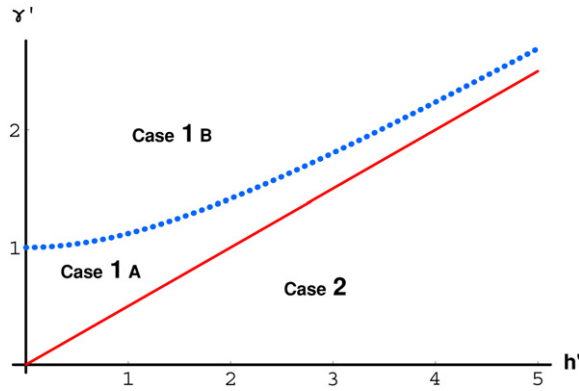
Figure A1. Phase diagram of the anisotropic XY model in a staggered magnetic field (only $\gamma' \geqslant 0$ and $h' \geqslant 0$ shown). The three cases 2, 1A, 1B, considered in this paper, are clearly marked. The critical phase ($h' = 2\gamma'$) is drawn as a bold line (red, online). The boundary between cases 1A and 1B, where the ground state is given by two degenerate product states, is shown as a dotted line (blue, online).

to calculate the entropy for a staggered field one only needs to take the appropriate result and perform the substitutions (A.2).

Using (A.2), the spectrum of the XY model in a staggered magnetic field is

$$\epsilon_k = \sqrt{(\gamma' \cos k - h'/2)^2 + \sin^2 k}. \tag{A.3}$$

From this, we see that the critical phase $h = 2$ is mapped to the line $h' = 2\gamma'$.

Therefore, the mapping of the different cases and the definitions of the elliptic parameter for this model are

$$\text{case 2} \qquad \{h' > 2\gamma': \qquad k \equiv 1/\sqrt{(h'/2) - \gamma'^2 + 1}, \tag{A.4}$$

$$\text{case 1A} \qquad \begin{cases} h' < 2\gamma', \\ \gamma' < \sqrt{1 + (h'/2)^2} \end{cases}: \qquad k \equiv \sqrt{(h'/2) - \gamma'^2 + 1}, \tag{A.5}$$

$$\text{case 1B} \qquad \begin{cases} h' < 2\gamma', \\ \gamma' > \sqrt{1 + (h'/2)^2} \end{cases}: \qquad k \equiv \frac{\sqrt{\gamma'^2 - (h'/2)^2 - 1}}{\sqrt{\gamma'^2 - (h'/2)^2}}. \tag{A.6}$$

We draw the phase diagram of the XY model in a staggered field and indicate the three cases in figure A1. With definitions (A.4)–(A.6), one can plug the elliptic parameter into (10)–(12) and use the other results of this paper to calculate the entropy of the XY model in a staggered magnetic field.

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